tion is an oval object, but what it is intended to represent I am at present unable to offer an explanation. The two last-described figures are shown in their relative positions to each other, but are not so in reference to the man. They are shown above the head of the latter on the plate to fill out a vacant space, but a careful reading of this description will indicate their true position.

ON THE TRANSITIVE SUBSTITUTION GROUPS THAT ARE SIMPLY ISOMORPHIC TO THE SYMMETRIC OR THE ALTERNATING GROUP OF DEGREE SIX.

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When the degree of a symmetric or an alternating group is not 6 we can obtain all the simple isomorphisms of the group to itself by transforming it by means of the substitutions of the symmetric group of the same degree. In other words, we can construct only one intransitive group of degree mn and order n ! or n ! ÷ 2, whose m transitive constituents are respectively the symmetric or the alternating group of degree n, $n \mp 6$.[1] Hence the number of transitive substitution groups that are simply isomorphic to the symmetric group of degree n ($n \pm 6$) is equal to the total number of substitution groups (transitive and intransitive) that can be constructed with n letters and whose order is less than n ! ÷ 2, while the number of those that are simply isomorphic to the alternating group of this degree is equal to the number of all the other positive groups that can be constructed with n letters.[2]

As nearly all the groups that can be constructed with n letters are subgroups of larger groups that do not involve the symmetric or the alternating group of degree n and whose degree $< n + 1$, the transitive substitution groups that are simply isomorphic to the

[1] Hölder, *Mathematische Annalen*, Vol. xlvi, pp. 340, 345; cf. Miller, *Bulletin of the American Mathematical Society* (1895), Vol. i, p. 258.

[2] Dyck, *Mathematische Annalen*, Vol. xxii, p. 90; cf. Miller, *Philosophical Magazine* (1897), Vol. xliii, p. 117.

symmetric or the alternating group of degree n are, as a rule, non-primitive.[1] Those that are simply isomorphic to the alternating group are simple when $n \mp 4$ and can therefore contain no substitution besides identity that leaves all of their systems of non-primitivity unchanged. The first group of this kind is of order 60 and degree 12. Its elements can be divided in only one way so that each division is a system of non-primitivity. Even when such a group has different sets of systems of non-primitivity it cannot contain any substitution besides identity that leaves all the systems of non-primitivity of any set unchanged since it contains no self-conjugate subgroup except identity.

The following well-known group illustrates that the elements of some non-primitive groups can be divided into different sets of systems of non-primitivity, so that the groups contain no substitution besides identity that leaves all the systems of one set unchanged while they contain other substitutions that leave those of another set unchanged.

1	ad. bf. ce
abc. def	ae. bd. cf
acb. dfe	af. be. cd

This group contains no substitution besides identity that leaves all the systems of non-primitivity of any one of the following three sets unchanged ;

a,d : b,e : c,f   a,e : b,f : c,d   a,f : b,d : c,e

while three of its substitutions leave the systems of the following set unchanged ;

a,b,c : d,e,f.

The other regular group of order 6 may serve to illustrate that the elements of some non-primitive groups cannot be divided into systems of non-primitivity so that the groups contain no substitution besides identity that leaves all the systems unchanged.

A non-primitive group that is simply isomorphic to the symmetric group of degree n ($n \mp 4$) contains only one selfconjugate subgroup besides identity. If its largest subgroup whose degree is less than the degree of the group corresponds to a positive subgroup of the symmetric group, its selfconjugate subgroup must be intransitive and it must have two systems of non-primitivity. If the given sub-

[1] Cf. Maillet, *Comptes Rendus*, Vol. cxix, p. 362.

group corresponds to a positive and negative subgroup of the symmetric group the selfconjugate subgroup of order n ! ÷ 2 is transitive, and the non-primitive group contains no substitution besides identity that leaves all the systems of any of the possible sets of systems of non-primitivity unchanged. We thus obtain composite non-primitive groups that do not contain any substitution besides identity that leaves all the systems of any of the possible sets of systems of non-primitivity unchanged. The first group of this kind is of order 120 and degree 15.

Both the symmetric and the alternating group of degree 6 contain 6 ! simple isomorphisms to themselves that cannot be obtained by transforming the groups by means of any substitutions whatever. If we regard these groups as operation groups of orders 6 ! and 6 ! ÷ 2 respectively we can find an operation group of order 2 × 6 ! which transforms both of them into all the possible simple isomorphisms to themselves.[1] This group is simply isomorphic to the group of all simple isomorphisms of the given operation groups to themselves and contains the symmetric group of order 6 ! as selfconjugate subgroup.

That the smallest operation group which transforms a given operation group into all the possible simple isomorphisms to itself is always simply isomorphic to the group of all these simple isomorphisms may be easily proved as follows. We represent the given operation group of order g by the regular substitution group. The largest group of degree g that contains it as a selfconjugate subgroup transforms it into all the possible simple isomorphisms to itself and has a g, 1 correspondence[2] to the group of these simple isomorphisms. Its subgroup which contains all the substitutions that do not involve one of its g elements must be simply isomorphic to the group of all the given simple isomorphisms since all the substitutions that are commutative to each of the substitutions of the given regular group form a regular group of degree g.

It is known that the symmetric or the alternating group of degree 6 can be made simply isomorphic to itself by placing the group of order 120 or 60 and degree 6 in correspondence to a group of degree 5. All the substitutions of order 3 in such a 1, 1 correspondence are of degree 9, for it is evident that some are of this degree and that all must have the same degree because all the subgroups of

[1] Cf. Frobenius, *Sitzungsberichte der Akademie zu Berlin*, 1895, p. 184.

[2] Jordan, *Traite des Substitutions*, p. 60.

order and degree 3 as well as those of order 3 and degree 6 are conjugate in the symmetric and in the alternating group. From this it follows that the substitutions of degree and order 2 must correspond to those of degree 6, and hence that all the substitutions of order 2 in such a 1, 1 correspondence are of degree 8.

The Fifteen Transitive Substitution Groups that are Simply Isomorphic to (abcdef) pos.

We shall consider these groups in the order of their degrees, beginning with those of the highest degrees. There is one group of degree 360, viz., the regular group. We represent it by G_1. As there is only one positive group of order[1] 2 and the two groups of order 3 can be made to correspond there is one transitive group (G_2) of degree 180 and one (G_3) of degree 120 that are simply isomorphic to (abcdef) pos. Since the given group of order 2 is transformed into itself by 8 positive substitutions and the given groups of order 3 are transformed into themselves by 18 positive substitutions G_2 and G_3 are respectively of class[2] 176 and 114.

There is only one positive cyclical group of order 4 and the two positive non-cyclical groups of this order can be made to correspond since the transitive one cannot occur in $(abcde)_{10}$, and hence it can also not occur in $(abcdef)_{60}$. The group (G_4) of degree 90 whose 4 substitutions that do not contain a given element form a cyclical group is of class 88 since there are only 8 positive substitutions that transform the given positive group of order 4, or its subgroup of order 2, into itself. The other group of this degree (G_5) is of class 84. In both of these groups, the subgroup which contains the four substitutions that do not involve a given element contains three substitutions of the same degree. Since there is only one group of order 5 there is only one group of degree 72 (G_6) that is simply isomorphic to (abcdef) pos. It is of class 70.

Since only one of the two positive groups of order 6 is found in (abcde) pos., they can be placed in correspondence and there is only one transitive group (G_7) of degree 60 that is simply to (abcdef)

[1] We consider only those groups whose degree does not exceed 6. A list of these groups is given by Prof. Cayley, *Quarterly Journal of Mathematics*, Vol. xxv, p. 71.

[2] The class of a substitution group is the degree of its substitution that permutes the smallest number of elements besides identity. Cf. Jordan, *Liouville's Journal*, 1871.

pos. We proceed to find the forms of the substitutions of the subgroup of G_7, which contains all its substitutions that do not involve a given element. Its substitutions of order 3 clearly consist of 19 distinct cycles of degree 3. If we arrange the substitutions of (abcdef) pos., according to [(abc) all (de)] pos., or (abc. def) all, we obtain 4 rows of substitutions whose — 1 powers transform any one of the substitutions of order two in either of these groups into substitutions of the same group. Hence the corresponding substitution of G_7 is of degree 56, and this is also the class of G_7. Its subgroup which includes the substitutions that do not involve a given element contains therefore 2 substitutions of order 3 and degree 57, 3 of order 2 and degree 56, and identity.

Since there is only one positive group of each of the orders 8, 9, 10 there is only one transitive group of each of the degrees 45, 40, 36 that is simply to (abcdef) pos. We shall denote these groups by G_8, G_9, G_{10}. Their classes are respectively 40, 36, 32. The two positive groups of order 12 lead to only one group (G_{11}) since one of them occurs in (abcde) pos. and its substitutions of order 3 are also of degree 3. G_{11} is of degree 30 and class 24. Half of its substitutions of order 3 are of degree 30 and the rest of degree 24. All its substitutions of order 2 are of degree 28. There can be only one transitive group of degree 20 (G_{12}) that is simply isomorphic to (abcdef) pos. since there is only one positive group of order 18. All of its 80 substitutions of order 3 are of degree 18 and its 45 substitutions of order 2 are of degree 16. As the other substitutions are of degree 20 G_{12} is of class 16.

There can be only one transitive group of degree 15 (G_{13}) that is simply isomorphic to (abcdef) pos. since the two positive groups of order 24 must evidently correspond in the simple isomorphism of (abcdef) pos. to itself in which all the substitutions of order 3 are of degree 9. G_{13} contains the following substitutions, besides identity, whose degrees are less than 15: 40 of order 3 and degree 12, 45 of order 2 and degree 12, 90 of order 4 and degree 14. Hence it is of class 12. The group of degree 10 (G_{14}) that depends upon the positive group of order 36 contains 90 substitutions of order 3 and degree 9, 45 of order 2 and degree 8, 90 of order 4 and degree 8. The rest are of degree 10 with the exception of identity. The group (G_{15}) which depends upon either of the two groups of order 60 is (abcdef) pos. itself.

Since each of the three groups

(abcde) pos. $\quad$ $(abcdef)_{36}$ $\quad$ [(abcd) all (ef)] pos.

is a maximal subgroup of (abcdef) pos. the corresponding groups, viz., G_{15}, G_{14}, G_{13} are primitive. The other 12 are non-primitive. As they are simple groups they cannot contain any substitution besides identity that leaves all the systems of non-primitivity unpermuted. If we arrange the substitutions of (abcdef) pos. according to (abcde) pos. we may letter the rows in such a manner that they are permuted according to a substitution that is identical to the particular substitution into which the entire group is multiplied. This is clearly not the case when they are arranged according to $(abcdef)_{60}$ since the necessary and sufficient condition that a given substitution leads to a substitution whose degree is less than the degree of the simply isomorphic transitive group is that it is conjugate to some substitution in the group which forms the first row. G_{13} is simply transitive since its order is not divisible by 14. G_{14} and G_{15} are clearly multiply transitive.

The Thirty-five Transitive Substitution Groups that are Simply Isomorphic to (abcdef) all.

The subgroups which correspond in a simple isomorphism of (abcdef) pos. to itself correspond also in some simple isomorphism of (abcdef) all to itself. Hence each group that leads to a transitive group of degree n that is simply isomorphic to (abcdef) pos. leads also to a transitive group of degree 2n that is simply isomorphic to (abcdef) all. We observed above that the latter groups contained two systems of non-primitivity and that they are the only groups which are simply isomorphic to (abcdef) all and have this property. Hence 15 of the non-primitive groups that are simply isomorphic to the symmetric group of degree 6 contain two systems of non-primitivity. Their degrees are as follows:

720, 360, 240, 180, 180, 144, 120, 90, 80, 72, 60, 40, 30, 20, 12.

The remaining 20 transitive groups that are simply isomorphic to (abcdef) all depend upon subgroups that involve negative substitutions. We shall therefore confine our attention to such subgroups. As the methods are similar to those employed in what precedes we shall be more brief in our explanations. The two groups of order 2 lead to the same transitive group (G_{16}) since all the substitutions of order 2 in some of the simple isomorphisms of (abcdef) all to itself are of degree 8, as observed above. G_{16} is of degree 360 and class 336.

The two groups of order 4

(ab) (cd) [(abcd)$_4$ (ef)] dim.

must correspond in some of the simple isomorphisms of (abcdef) all to itself for the reason just given. They therefore lead to the same group of degree 180 (G_{17}). Since the largest group that transforms (ab. cd) (ef) into itself must also transform each of its substitutions into itself this group could not correspond to either of the preceding groups in any of the given isomorphisms. It therefore leads to a different group of degree 180 (G_{18}). It is evident that (abcd) cyc. leads to another group of this degree (G_{19}). The two cyclical as well as the two non-cyclical groups of order 6 must correspond in the simple isomorphism of (abcdef) all to itself in which the substitutions of order 3 are of degree 9. Hence these lead to only two groups of degree 120 (G_{20}, G_{21}).

The two groups of order 8 which contain no substitution whose order exceeds 2 must correspond in the isomorphisms in which the substitutions of order 2 are of degree 8. Hence they lead to the same group of degree 90 (G_{22}). In the same isomorphisms (abcd)$_8$ must correspond to [(abcd)$_8$ (ef)] dim. with respect to (abcd) cyc. We represent the group which depends upon either of these two groups by G_{23}. Each of the remaining two groups of order 8

(abcd) cyc. (ef), [(abcd)$_8$ (ef)] dimidiated with respect to (ab) (cd)

leads to a transitive group of degree 90 that is simply isomorphic to (abcdef) all. We represent these two groups by G_{24} and G_{25} respectively. The former of the given groups of order 8 is the only one that is commutative and contains operations of order 4 and the latter is the only one that is non-commutative and contains operations of order 4 and degree 6. On account of these properties each of these groups must correspond to itself in all the possible simple isomorphisms of (abcdef) all to itself.

The two groups of order 12 must correspond in some of these simple isomorphisms since the substitutions of order 3 are of degree 3 in the one and of degree 6 in the other. Hence they lead to the same group of degree 60 (G_{26}). As there is only one group of order 16 there is only one group of degree 45 (G_{27}). Since the subgroups of degree 6 and order 3 in $(abcdef)_{18}$ are self-conjugate while those of degree and order 3 are conjugate, this group must correspond to the other group of order 18 in the isomorphism in which all the substitutions of order 3 are of degree 9. Hence there is only one group of degree 40 (G_{28}). The single group of order 20 leads to a group of degree 36 (G_{29}). It is evident that (abcd) all and $(\pm\ abcdef)_{24}$ as well as (abcd) pos. (ef) and $(abcdef)_{24}$, lead to the same group. These two groups (G_{30} and G_{31}) are of degree 30.

The transitive and the intransitive group of order 36 lead to the same group of degree 20 (G_{32}) for the same reason as was employed to show that the transitive and the intransitive group of order 18 lead to the same group. The groups which we employed so far are non-maximal subgroups of (abcdef) all. Hence each of the 32 groups given above is non-primitive. Only the first fifteen of them contain substitutions which leave all the systems of non-primitivity unchanged. It remains to find the three primitive groups that are simply isomorphic to (abcdef) all.

The one of the highest degree (G_{33}) depends upon either one of the following groups

$$(abcdef)_{48} \qquad (abcd)\ \text{all}\ (ef)$$

It is therefore of degree 15. The next (G_{34}) depends upon the single group of order 72 and hence is of degree 10. The last (G_{35}) depends upon (abcde) pos. or upon $(abcdef)_{120}$. This is (abcdef) all itself. G_{33} is simply transitive since its order is not divisible by 14. As it contains 48 substitutions that transform each of the 7 substitutions of the form ab. cd. ef. contained in $(abcdef)_{48}$ into itself it is of class 8. We have already observed that the self-conjugate subgroups of G_{34} and G_{35} are multiply transitive, hence the groups themselves must be multiply transitive. Their classes are 6 and 2 respectively.

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